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CERTAIN FACTORS OF THE GROUP DETERMINANT.

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So far as I know the following method for finding factors of a group determinant is new.

To the n operators $s_1, s_2, \ldots, s_n$ of a group G of finite order we make correspond n independent variables $x_{s_1}, x_{s_2}, \ldots, x_{s_n}$, such that $x_{s_k} \equiv x_{s_i s_j}$ if $s_k = s_i s_j$. With these variables we form the following determinant, which is called the group determinant of G:

$$\Theta(x) \equiv \begin{vmatrix} x_{s_1 s_1^{-1}} & x_{s_1 s_2^{-1}} & \cdots\cdots & x_{s_1 s_n^{-1}} \\ x_{s_2 s_1^{-1}} & x_{s_2 s_2^{-1}} & \cdots\cdots & x_{s_2 s_n^{-1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ x_{s_n s_1^{-1}} & x_{s_n s_2^{-1}} & \cdots\cdots & x_{s_n s_n^{-1}} \end{vmatrix}$$

If we take for G the symmetric group of degree 3 and put $s_1 \equiv 1$, $s_2 \equiv (abc)$, $s_3 \equiv (acb)$, $s_4 \equiv (ab)$, $s_5 \equiv (bc)$, $s_6 \equiv (ac)$, then

$$\Theta(x) \equiv \begin{vmatrix} x_{s_1} & x_{s_3} & x_{s_2} & x_{s_4} & x_{s_5} & x_{s_6} \\ x_{s_2} & x_{s_1} & x_{s_3} & x_{s_5} & x_{s_6} & x_{s_4} \\ x_{s_3} & x_{s_2} & x_{s_1} & x_{s_6} & x_{s_4} & x_{s_5} \\ x_{s_4} & x_{s_5} & x_{s_6} & x_{s_1} & x_{s_3} & x_{s_2} \\ x_{s_5} & x_{s_6} & x_{s_4} & x_{s_2} & x_{s_1} & x_{s_3} \\ x_{s_6} & x_{s_4} & x_{s_5} & x_{s_3} & x_{s_2} & x_{s_1} \end{vmatrix}$$

Let H, of order h, be any subgroup of G, and arrange the elements of the first column of $\Theta(x)$ with respect to the divisions $s_i H$. Then $\Theta(x)$ can be divided up into squares of h elements each, such that in any square the sums of the elements of the h rows are the same. For, the subscripts of the elements in the square containing the ith row and the jth column are contained in $s_i H . H s_j^{-1}$, the subscripts for the different rows being obtained by taking the different operators in the first H, and those for the different columns by taking the different operators in the second H.

We now form the determinant Θ_1, of degree n/h, by taking for each element the sum of the elements in any row of the corresponding square of $\Theta(x)$. If we arrange the elements of the determinant of the symmetric group of degree 3 with respect to the subgroup $[1, (ab)]$, the corresponding Θ_1 is

$$\begin{vmatrix} x_{s_1}+x_{s_4} & x_{s_3}+x_{s_5} & x_{s_2}+x_{s_6} \\ x_{s_2}+x_{s_5} & x_{s_1}+x_{s_6} & x_{s_3}+x_{s_4} \\ x_{s_3}+x_{s_6} & x_{s_2}+x_{s_4} & x_{s_1}+x_{s_5} \end{vmatrix}$$

For convenience I shall hereafter omit the x's and write only the subscripts.

I wish to prove that Θ_1 *is a factor of* Θ. (a) Add the elements of the first h rows of Θ to the corresponding elements of the first row; the elements of the second h rows to the corresponding elements of the $(h+1)$th row; and so on. (b) Now so re-arrange the rows and columns that the minor formed by the first h rows and columns is Θ_1. (c) Then multiply the elements of every row, except the first h rows, by h. (d) After this, subtract the elements of the first row from the corresponding elements of each row that was originally among the first h rows; the elements of the second row from the corresponding elements of each row that was originally among the second h rows; and so on. (e) Now to the elements of the first column add the corresponding elements of the columns that were originally the first h columns; to the elements of the second column add the corresponding elements of the columns that were originally among the second h columns; and so on. In the final form of the determinant each element of the first diagonal square of order h is h times the corresponding element of Θ_1 and all the other elements of the first h columns are zero. Moreover, a further obvious modification shows that the remaining elements of the first h rows can all be made zero. Therefore Θ_1 is a factor of Θ.

I shall illustrate the steps of this proof by the determinant of the symmetric group of degree 3.

$$\Theta \equiv \begin{vmatrix} s_1 & s_4 & s_3 & s_5 & s_2 & s_6 \\ s_4 & s_1 & s_5 & s_3 & s_6 & s_2 \\ s_2 & s_5 & s_1 & s_6 & s_3 & s_4 \\ s_5 & s_2 & s_6 & s_1 & s_4 & s_3 \\ s_3 & s_6 & s_2 & s_4 & s_1 & s_5 \\ s_6 & s_3 & s_4 & s_2 & s_5 & s_1 \end{vmatrix}$$

$$
\text{(a)}\quad \begin{vmatrix}
s_1+s_4 & s_4+s_1 & s_3+s_5 & s_5+s_3 & s_2+s_6 & s_6+s_2 \\
s_4 & s_1 & s_5 & s_3 & s_6 & s_2 \\
s_2+s_5 & s_5+s_2 & s_1+s_6 & s_6+s_1 & s_3+s_4 & s_4+s_3 \\
s_5 & s_2 & s_6 & s_1 & s_4 & s_3 \\
s_3+s_6 & s_6+s_3 & s_2+s_4 & s_4+s_2 & s_1+s_5 & s_5+s_1 \\
s_6 & s_3 & s_4 & s_2 & s_5 & s_1
\end{vmatrix}
$$

$$
\text{(b), (c)}\quad \begin{vmatrix}
s_1+s_4 & s_3+s_5 & s_2+s_6 & s_4+s_1 & s_5+s_3 & s_6+s_2 \\
s_2+s_5 & s_1+s_6 & s_3+s_4 & s_5+s_2 & s_6+s_1 & s_4+s_3 \\
s_3+s_6 & s_2+s_4 & s_1+s_5 & s_6+s_3 & s_4+s_2 & s_5+s_1 \\
2s_4 & 2s_5 & 2s_6 & 2s_1 & 2s_3 & 2s_2 \\
2s_5 & 2s_6 & 2s_4 & 2s_2 & 2s_1 & 2s_3 \\
2s_6 & 2s_4 & 2s_5 & 2s_3 & 2s_2 & 2s_1
\end{vmatrix}
$$

$$
\text{(d)}\quad \begin{vmatrix}
s_1+s_4 & s_3+s_5 & s_2+s_6 & s_4+s_1 & s_5+s_3 & s_6+s_2 \\
s_2+s_5 & s_1+s_6 & s_3+s_4 & s_5+s_2 & s_6+s_1 & s_4+s_3 \\
s_3+s_6 & s_2+s_4 & s_1+s_5 & s_6+s_3 & s_4+s_2 & s_5+s_1 \\
s_4-s_1 & s_5-s_3 & s_6-s_2 & s_1-s_4 & s_3-s_5 & s_2-s_6 \\
s_5-s_2 & s_6-s_1 & s_4-s_3 & s_2-s_5 & s_1-s_6 & s_3-s_4 \\
s_6-s_3 & s_4-s_2 & s_5-s_1 & s_3-s_6 & s_2-s_4 & s_1-s_5
\end{vmatrix}
$$

$$
\text{(e)}\quad \begin{vmatrix}
2(s_1+s_4) & 2(s_3+s_5) & 2(s_2+s_6) & s_4+s_1 & s_5+s_3 & s_6+s_2 \\
2(s_2+s_5) & 2(s_1+s_6) & 2(s_3+s_4) & s_5+s_2 & s_6+s_1 & s_4+s_3 \\
2(s_3+s_6) & 2(s_2+s_4) & 2(s_1+s_5) & s_6+s_3 & s_4+s_2 & s_5+s_1 \\
0 & 0 & 0 & s_1-s_4 & s_3-s_5 & s_2-s_6 \\
0 & 0 & 0 & s_2-s_5 & s_1-s_6 & s_3-s_4 \\
0 & 0 & 0 & s_3-s_6 & s_2-s_4 & s_1-s_5
\end{vmatrix}
$$

If H' is conjugate to H in G, the corresponding factor Θ'_1 is the same as Θ_1. For, the element in the ith row and jth column of Θ_1 is Σs_α, where s_α runs through the operators of $s_i H s_j^{-1}$. If $s_r H s_r^{-1} = H'$, the element in the ith row and jth column of Θ_1 is the same as the element in the kth row and lth column of Θ'_1, where $s_i = s_k s_r$, $s_j^{-1} = s_r^{-1} s_l^{-1}$, since $s_i H s_j^{-1} = s_k s_r H s_r^{-1} s_l^{-1} = s_k H' s_l^{-1}$. Moreover, s_j fixes s_l, and therefore the elements of the jth column of Θ_1 are the same, except as to arrangement, as those of the lth column of Θ'_1. Likewise the elements of the ith row of Θ_1 are the same, except as to arrangement, as those of the kth row of Θ'_1. Therefore $\Theta'_1 = \pm\Theta_1$.